# Module 5: Conic Sections and Polar Coordinates

### II. The Circle and the Ellipse

After completing this section, you should be able to:

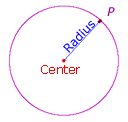
* identify the center and radius of a circle, given its equation
* identify the center, vertices, foci, major axis, and minor axis of an ellipse, given its equation
* transform an equation of a circle or an ellipse into standard form
* graph circles and ellipses

#### A. The Circle

# Geometric Definition of a Circle

A **circle** is the set of all points in a plane that are located a fixed distance from a fixed point (the **center**) in the plane.

A **radius** of a circle is a line segment joining the center and a point on the circle. The fixed distance between the center and a point on the circle is the length of the radius.



Graphs of circles were reviewed in module 1, topic I. Recall the standard equation of a circle:

|  |  |  |
| --- | --- | --- |
| |  | | --- | | Standard Equation of a Circle The equation of a circle with center (h, k) and radius r, in **standard form**, is  (x – h)2 + (y – k)2 = r2 | | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/standard-eq-circle.png |

A general equation of a circle can be transformed to standard form by completing the square.

**Example II.A.1:** Find the center and the radius of the circle given by the general equation.

3x2 + 3y2 + 12x – 36y – 27 = 0

Solution:

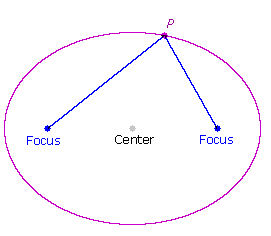
|  |  |
| --- | --- |
| 3x2 + 3y2 + 12x – 36y – 27 = 0 |  |
| (3x2 + 12x) + (3y2 – 36y) = 27 | Group terms. In order to apply the technique of completing the square, the coefficients of the x2 and y2terms must be equal to 1. |
| 3[(x2 + 4x) + (y2 – 12y)] = 27 | Factor out 3. |
| (x2 + 4x) + (y2 – 12y) = 9 | Divide both sides by 3. |
| (x2 + 4x + 4 – 4) + (y2 – 12y + 36 – 36) = 9 | Complete the square. |
| (x2 + 4x + 4) + (y2 – 12y + 36) – 4 – 36 = 9 | Regroup. |
| (x + 2)2 + (y – 6)2 = 49 | Factor and simplify. |
| [x – (–2)]2 + (y – 6)2 = 72 | Rewrite in standard form. |
| The center is (–2, 6) and the radius is 7.  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/mod5-exII-a1.png | |

As illustrated by the example, a general second-degree equation of a circle must have an x2 term and a y2 term, and the coefficients of those terms must be equal.

#### B. The Ellipse

# Geometric Definition of an Ellipse

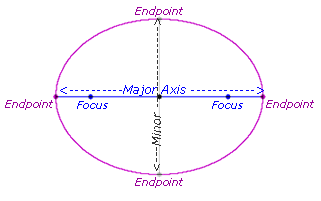
An **ellipse** is the set of all points in the plane for which the sum of the distances from two fixed points (**foci**) is constant.



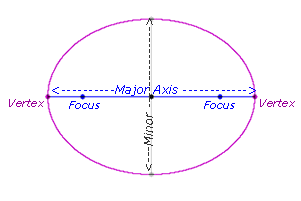
The **center** of an ellipse is the midpoint of the line segment between the foci.

You can [bring the definition to life](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/popups/Ellipse-pushpins.html) by tracing an ellipse using a piece a piece of string, two pushpins (tacks), a piece of cardboard, and a pen.

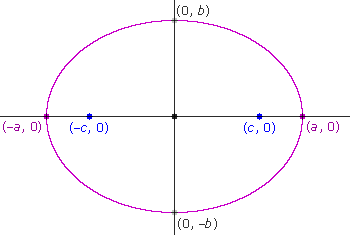
An ellipse has two perpendicular axes of symmetry, which intersect at the center of the ellipse. The **endpoints** of the ellipse are the intersection points of the lines of symmetry and the ellipse. The **major axis** is the line segment that joins two endpoints and passes through the foci. The **minor axis** is the line segment joining the other two endpoints.



The endpoints of the major axis are called **vertices**:



##### Standard Equation of an Ellipse with Center (0, 0)



Consider the case in which the center of the ellipse is the origin (0, 0) and the foci (–c, 0) and (c, 0) are located on the x-axis, so the major axis is horizontal.

Label the vertices (–a, 0) and (a, 0). The length of the major axis is 2a.

Label the endpoints of the minor axis (0, –b) and (0, b). The length of the minor axis is 2b.

Using this labeling and the geometric definition of an ellipse, it can be shown that

https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/ellipse-formula.gif

This is the standard form of an ellipse with center (0, 0) and major axis on the x-axis.

Furthermore, it can be shown that the major axis is longer than the minor axis (so a > b), and the coordinates of the endpoints and foci are related by the equation c2 = a2 – b2.

You can also start with the standard equation of an ellipse and find the foci.

**Example II.B.1:** Given the following equation of an ellipse, find the foci and graph the ellipse

x2 + 4y2 – 4y = 0

Solution:

|  |  |
| --- | --- |
| x2 + 4y2 – 4 = 0 |  |
| x2 + 4y2 = 4 | Add 4 to both sides. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-1.gif | Divide both sides by 4 to get 1 on the right side. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-1a.gif | Rewrite the equation in standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/ellipse-formula-red.gif. |

a = 2 and b = 1. The major axis is on the x-axis.

The center of the ellipse is (0, 0).

The vertices are (–a, 0) = (–2, 0) and (a, 0) = (2, 0).

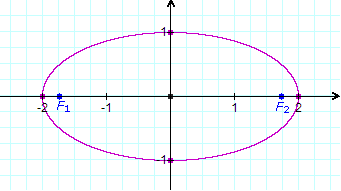
The endpoints of the minor axis are (0, –b) = (0, –1) and (0, b) = (0, 1).

The foci have the form (–c, 0) and (c, 0), where c2 = a2 – b2 = (2)2 – (1)2 = 3.

Because c2 = 3, c =square root of 3.

The foci are F1: (–c, 0) = (–square root of 3, 0) and F2: (c, 0) = (square root of 3, 0).

The graph is shown below.



If the center of the ellipse is (0, 0) and the major axis is the y-axis (rather than the x-axis), interchange the roles of x and y to arrive at the standard equation of the ellipse.

Here is a summary of the standard equations and properties of ellipses with center (0, 0).

|  |
| --- |
| **Standard Equation of an Ellipse with Center (0, 0)** |
| **Major Axis on the x-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/ellipse-formula.gif, where a > b > 0  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/ellipse-w-center.png   |  |  |  |  | | --- | --- | --- | --- | | Axis | Orientation | Length | Selected Points on Axis | | Major | Horizontal | 2a | Vertices (–a, 0) and (a, 0)  Foci (–c, 0) and (c, 0), where c2 = a2 – b2 | | Minor | Vertical | 2b | Endpoints (0, –b) and (0, b) | |
| **Major Axis on the y-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/maj-axis-on-y-axis.gif, where a > b > 0  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/ellipse-w-centr-yaxis.png   |  |  |  |  | | --- | --- | --- | --- | | Axis | Orientation | Length | Selected Points on Axis | | Major | Vertical | 2a | Vertices (0, –a) and (0, a)  Foci (0, –c) and (0, c), where c2 = a2 – b2 | | Minor | Horizontal | 2b | Endpoints (–b, 0) and (b, 0) | |

**Caution**: For both of these standard forms, 2a is the length of the major axis and 2b is the length of the minor axis. Although x and y are interchanged in the two forms, a and b are not. The value of a is always greater than the value of b.

**Example II.B.2:** Find the standard equation of an ellipse having vertices (0, –5,) and (0, 5), and foci (0, –4) and (0, 4). Determine the lengths of the major and minor axes. Graph the ellipse.

Solution:

The center of the ellipse is the midpoint of the major axis (which is also the midpoint of the line segment between the foci). Since the vertices are (0, –5,) and (0, 5), the center is the midpoint (0, 0), and also the major axis is vertical. Therefore the standard equation has the form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/maj-axis-on-y-axis.gif.

The vertices have the form V1: (0, –a) = (0, –5) and V2: (0, a) = (0, 5), so a = 5.

The foci have the form F1: (0, –c) = (0, –4) and F2: (0, c) = (0, 4), so c = 4.

Use the equation c2 = a2 – b2 to find b:

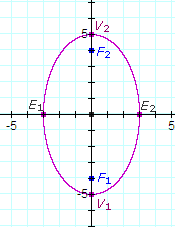
 c2 = a2 – b2   
(4)2 = (5)2 – b2   
16 = 25 – b2   
9 = b2   
b = 3

The equation of the ellipse is https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-2.gif, or, equivalently, https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-2a.gif.

The length of the major axis is 2a = 2(5) = 10 and the length of the minor axis is 2b = 2(3) = 6.

The endpoints of the minor axis are E1: (–b, 0) = (–3, 0) and E2: (b, 0) = (3, 0).

The graph is shown below.



Now suppose the center of an ellipse is (h, k) rather than (0, 0). Just as in the case of the parabola, going from (0, 0) to (h, k) entails a horizontal shift of h units and a vertical shift of k units. The foci, vertices, and endpoints are shifted as well. Start with a standard equation of an ellipse with center (0, 0) and replace x with x – h and y with y – k to arrive at the standard form of an ellipse with center (h, k). The results are summarized below.

|  |
| --- |
| **Standard Equation of an Ellipse with Center (h, k)** |
| **Major Axis Parallel to the x-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/standard-formla-4-ellipse.gif, where a > b > 0  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/ellipse-w-center2.png   |  |  |  |  | | --- | --- | --- | --- | | Axis | Orientation | Length | Selected Points on Axis | | Major | Horizontal | 2a | Vertices (h – a, k) and (h + a, k)  Foci (h – c, k) and (h + c, k), where c2 = a2 – b2 | | Minor | Vertical | 2b | Endpoints (h, k – b) and (h, k + b) | |
| **Major Axis Parallel to the y-Axis** |
| Standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/standard-formula-4-ellipse.gif, where where a > b > 0  https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/Graphs/ellipse-w-centr-yaxis2.png   |  |  |  |  | | --- | --- | --- | --- | | Axis | Orientation | Length | Selected Points on Axis | | Major | Vertical | 2a | Vertices (h, k – a) and (h, k + a)  Foci (h, k – c) and (h, k + c), where c2 = a2 – b2 | | Minor | Horizontal | 2b | Endpoints (h – b, k) and (h + b, k) | |

If you start with an equation of an ellipse that is not in standard form, you can transform it to standard form by completing the square. (See module 1, topic I-C to review this technique.)

**Example II.B.3:** Find the center, vertices, and foci of the ellipse given by the general equation

4x2 + 9y2 – 32x + 18y + 37 = 0

and graph the ellipse.

Solution:

|  |  |
| --- | --- |
| 4x2 + 9y2 – 32x + 18y + 37 = 0 |  |
| (4x2 – 32x) + (9y2 + 18y) + 37 = 0 | Group terms. |
| 4(x2 – 8x) + 9(y2 + 2y) + 37 = 0 | Factor. |
| 4(x2 – 8x + 16 – 16) + 9(y2 + 2y + 1 – 1) + 37 = 0 | Complete the square. |
| 4(x2 – 8x + 16) – 64 + 9(y2 + 2y + 1) – 9 + 37 = 0 | Multiply and regroup, using the distributive property. |
| 4(x – 4)2 + 9(y + 1)2 – 36 = 0 | Factor and simplify. |
| 4(x – 4)2 + 9(y + 1)2 = 36 | Rewrite in standard form. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-3.gif | Divide by 36 to get 1 on the right side. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-3a.gif | Simplify. |
| https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/mod5-II-B-3b.gif | Rewrite in standard form https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M5-Module_5/images/standard-formla-4-ellipse-red.gif, where a > b > 0. |

The center of the ellipse is the point (4, –1). Because 3 > 2, a = 3 and b = 2, and the major axis is horizontal.

The vertices are located 3 units to the left and the right of the center (4, –1):

          V1: (4 – 3, –1) = (1, –1) and V2: (4 + 3, –1) = (7, –1).

The endpoints of the minor axis are located 2 units above and below the center (4, –1):

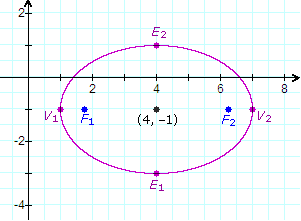
          E2: (4, –1 – 2) = (4, –3) and E2: (4, –1 + 2) = (4, 1).

Find c:

c2 = a2 – b2  
c2 = (3)2 – (2)2  
c2 = 9 – 4  
c2 = 5  
c = square root of 5

The foci are located square root of 5 units to the left and the right of the center (4, –1):

F1: (4 –square root of 5, –1) and F2: (4 +square root of 5, –1)



As illustrated by the example, a general second-degree equation of an ellipse must have an x2 term and a y2 term, and the coefficients of those terms are unequal but of the same sign (both positive or both negative). If an equation of a conic section has x2 and y2 terms with coefficients of opposite signs, then the conic section is a hyperbola (to be studied in the next topic).

Recall from the ellipse tracings involving the pushpins that the closer together the foci, the closer the shape is to a circle. Recall that c is the distance between a focus and the center of the ellipse, and a is the distance between a vertex and the center.

The **eccentricity** e of an ellipse is given by the formula e = c/a. It measures the roundness of an ellipse.

Since 0 < c < a, then 0 < c/a < 1, and so the eccentricity is a number between 0 and 1.

The more round the ellipse, the closer the foci are to the center of the ellipse, the closer c is to 0, and the closer the eccentricity e is to 0. The more elongated the ellipse, the closer the foci are to the vertices of the ellipse, and the closer e is to 1.

The ellipse in example II.B.3 has eccentricity Eccentricity Equation.

Ellipses are at the heart of the mathematical model of the solar system formulated by Johannes Kepler in the early 1600s. Kepler proposed three laws of planetary motion. The first law states that the orbit of a planet about the sun is an ellipse, with the sun at one focus.

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